Bosonization of
Admissible Representations of $U_q(\widehat{sl}_2)$
at Level $-\frac{1}{2}$ and q-Vertex Operators

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Abstract

We construct a representation of $U_q(\widehat{sl}_2)$ at level $-1/2$ by using the bosonic Fock spaces. The irreducible modules are obtained as the kernel of a certain operator, in contrast to the construction by Feingold and Frenkel for $q = 1$ where such a procedure is not necessary. We also bosonize the q-vertex operators associated with the vector representation.

1   Introduction

The q-vertex operators introduced by Frenkel and Reshetikhin([7]) have important roles in the theory of solvable lattice models. In particular, at a positive integral level, the q-vertex operators are closely related to generalizations of the XXZ spin-chain and the RSOS models.([3],[4],[8])

In [12], Kac and Wakimoto introduced a class of irreducible highest weight representations of affine Lie algebras called admissible representations which is wider than the class of integrable representations. Admissible representations have a fractional level, and therefore they are not integrable in general. They proved the character formula for admissible representations, which is a generalization of the Weyl-Kac character formula. From this formula, we see

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that the set of the characters of admissible representations of the same level is closed under the modular transformations. Furthermore, the minimal series of the Virasoro algebra are constructed from the admissible representations by tensoring the level 1 integrable representation(the coset construction). This shows that the concept of the admissible representations is a natural generalization of that of the integrable representations.

Feingold and Frenkel constructed level $-1/2$ highest weight admissible representations of $\widehat{sp}_{2n}$ in terms of bosons([5]). In $\widehat{sp}_2 = \widehat{sl}_2$ case, the representations are constructed as follows. Let $\{\beta_n, \gamma_n | n \in I\}$ ($I = \mathbf{Z}$ or $\mathbf{Z} + 1/2$) be the following bosonic β-γ system

$$[\beta_m, \beta_n] = [\gamma_m, \gamma_n] = 0, \quad [\beta_m, \gamma_n] = \delta_{m+n,0},$$

and let $\mathcal{F}$ be the Fock space generated by the vacuum vector $|vac\rangle$ satisfying

$$\beta_m|vac\rangle = \gamma_n|vac\rangle = 0 \quad \text{for } m \geq 0, n > 0.$$

We set

$$E(n) = -\frac{1}{2} \sum_{n=m+m'} \beta_m \beta_{m'},$$

$$H(n) = - \sum_{n=m+m'} : \beta_m \gamma_{m'} :,$$

$$F(n) = \frac{1}{2} \sum_{n=m+m'} \gamma_m \gamma_{m'},$$

where $:\quad:$ is the normal ordering defined by

$$:\beta_m \gamma_n := \begin{cases} \beta_m \gamma_n & (m < n) \\ \frac{1}{2}(\beta_m \gamma_n + \gamma_n \beta_m) & (m = n) \\ \gamma_n \beta_m & (m > n) \end{cases}$$

Then, $\{E(n), H(n), F(n) | n \in \mathbf{Z}\}$ forms $\widehat{sl}_2$ of level $-1/2$ and $\mathcal{F}$ is completely reducible as an $\widehat{sl}_2$-module. The decomposition of $\mathcal{F}$ into the irreducible components is calculated by characters([14]). In [5], the decomposition was proved by using the action of the Virasoro algebra via the Sugawara construction.

Let $\phi_1(z)$, $\phi_2(z)$ be two independent bosonic fields normalized by

$$\phi_1(z)\phi_1(w) \sim -\log(z - w),$$

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$$\phi_2(z)\phi_2(w) \sim \log(z - w).$$

It is known that the bosonic β-γ system can be expressed by $\phi_1(z)$ and $\phi_2(z)$ as follows([6]).

$$\beta(z) = \sum_{n \in I} \beta_n z^{-n-1} \longrightarrow\ :\partial_z\phi_2(z)e^{\phi_1(z)+\phi_2(z)}:\ ,$$

$$\gamma(z) = \sum_{n \in I} \gamma_n z^{-n} \longrightarrow\ :e^{-\phi_1(z)-\phi_2(z)}:\ .$$

The Fock space of the bosonic β-γ system is obtained as $\operatorname{Ker} Q^-$, where $Q^- = \oint\ :e^{-\phi_2(z)}:$. Using the above picture, we can rewrite the representation by Feingold and Frenkel in terms of $\phi_1(z)$, $\phi_2(z)$.

The aim of this paper is to construct a q-analogue of the representation in terms of $\phi_1(z)$, $\phi_2(z)$ and bosonize the corresponding q-vertex operators associated with the vector representation (cf.[16]). In Section 2, we review the known results about the quantum affine algebras and admissible representations. In Section 3, we construct the q-analogue of the above representation. In Section 4, we bosonize the q-vertex operators. Section 5 is devoted to the proofs which are not given in the previous sections.

2 Preliminaries

In this section, we recall the definition of the quantum affine algebra $U_q(\widehat{sl}_2)$ in order to fix the notations and review the results about the admissible representations in [12].

Let P and P^* be free $\mathbf{Z}$-modules as follows.

$$P := \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\delta\ , \quad P^* := \operatorname{Hom}(P, \mathbf{Z}) = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \mathbf{Z}d\ .$$

We call P and P^* the weight lattice and the coroot lattice respectively. The pairing is given by $\langle \Lambda_i, h_j \rangle = \delta_{ij}$, $\langle \Lambda_i, d \rangle = 0$, $\langle \delta, h_j \rangle = 0$, $\langle \delta, d \rangle = 1$. Let $\alpha_0 = 2\Lambda_0 - 2\Lambda_1 + \delta$, $\alpha_1 = 2\Lambda_1 - 2\Lambda_0$ be the simple roots. We fix an invariant symmetric bilinear form on P such that $(\alpha_i|\alpha_i) = 2$, $(\alpha_0|\alpha_1) = -2$.

We denote the base field $\mathbf{Q}(q^{\frac{1}{4}})$ by $\mathbf{K}$. The quantum affine algebra $U_q(\widehat{sl}_2)$ is the $\mathbf{K}$-algebra generated by the symbols $\{\gamma^{\pm\frac{1}{2}}, K, q^{\pm d}, h_k, x_l^{\pm}\ (k \in \mathbf{Z} \setminus \{0\}, l \in \mathbf{Z})\}$ which satisfy the following defining relations([2]).

$$\gamma^{\pm\frac{1}{2}} \in \textit{Center of } U_q(\widehat{sl}_2)\ , \quad \gamma^{\frac{1}{2}}\gamma^{-\frac{1}{2}} = 1\ , \quad KK^{-1} = K^{-1}K = 1\ ,$$

$$q^d q^{-d} = q^{-d} q^d = 1 \ , \quad q^d x_k^\pm q^{-d} = q^k x_k^\pm \ , \quad q^d h_k q^{-d} = q^k h_k \ , \quad q^d K = K q^k \ ,$$

$$[h_k, K] = 0 \ , \quad [h_k, h_l] = \delta_{k+l,0} \frac{[2k]}{k} \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}} \ , \quad [h_k, x_l^\pm] = \pm \frac{[2k]}{k} \gamma^{\mp \frac{|k|}{2}} x_{k+l}^\pm \ ,$$

$$K x_k^\pm K^{-1} = q^{\pm 2} x_k^\pm \ , \quad x_{k+1}^\pm x_l^\pm - q^{\pm 2} x_l^\pm x_{k+1}^\pm = q^{\pm 2} x_k^\pm x_{l+1}^\pm - x_{l+1}^\pm x_k^\pm \ ,$$

$$[x_k^+, x_l^-] = \frac{1}{q - q^{-1}} (\gamma^{(k-l)/2} \psi_{k+l} - \gamma^{(l-k)/2} \varphi_{k+l})$$

where

$$\sum_{k=0}^{\infty} \psi_k z^{-k} = K \exp\left((q - q^{-1}) \sum_{k=1}^{\infty} a_k z^{-k} \right) \ ,$$

$$\sum_{k=0}^{\infty} \varphi_{-k} z^{-k} = K^{-1} \exp\left(-(q - q^{-1}) \sum_{k=1}^{\infty} a_{-k} z^{-k} \right) \ .$$

Here we used the standard notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The Chevalley generators $\{e_i, f_i, t_i\}$ are given by

$$t_0 = \gamma K^{-1}, \ e_0 = x_1^- K^{-1}, \ f_0 = K_1 x_{-1}^+, \ t_1 = K, \ e_1 = x_0^+, \ f_1 = x_0^-.$$

The algebra $U_q(\widehat{sl}_2)$ has a Hopf algebra structure with the coproduct Δ and the antipode S given as follows.

$$\Delta(t_i) = t_i \otimes t_i \ , \quad \Delta(q^d) = q^d \otimes q^d \ ,$$

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i \ , \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i \ ,$$

$$S(t_i) = t_i^{-1} \ , \ S(q^d) = q^{-d} \ , \ S(e_i) = -t_i^{-1} e_i \ , \ S(f_i) = -f_i t_i \ .$$

Throughout this paper, we denote $U_q(\widehat{sl}_2)$ by U_q and the subalgebra of U_q generated by $\{t_i^{\pm 1}, e_i, f_i \ (i = 0, 1)\}$ by U_q'. We denote by $V(\lambda)$ the irreducible highest weight U_q(or U_q')-module with highest weight λ. We fix a highest weight vector of $V(\lambda)$ and denote it by $|\lambda\rangle$. If $W_i \ (i = 1, 2)$ has a weight decomposition $W_i = \oplus_\mu W_{i,\mu}$, we define their completed tensor product by

$$W_1 \widehat{\otimes} W_2 = \bigoplus_\mu \left(\prod_{\mu = \mu_1 + \mu_2} W_{1,\mu_1} \otimes W_{2,\mu_2} \right).$$

Admissible representations are irreducible highest weight representations whose highest weight is an admissible weight([12]). Admissible weights are defined as follows.

Definition.([12],[13]) *We call a weight λ admissible if it satisfies the following two conditions.*

1) $\langle \lambda + \rho, \alpha^\vee \rangle \notin \{0, -1, -2, -3, \cdots\}$ *for all real positive coroots $\alpha^\vee$,*

2) $\mathbf{Q}R^\lambda = \mathbf{Q}\Pi^\vee$

where ρ is the sum of all fundamental weights, $\Pi^\vee$ is the set of simple coroots and

$$R^\lambda = \{\alpha^\vee : a \text{ positive real coroot} | \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbf{Z}\}.$$

$\square$

The classification of admissible weights is given in [13]. In the $A_1^{(1)}$ case, the set of admissible weights of level $m = t/u$ $(t, u \in \mathbf{Z}, (t, u) = 1, u \geq 1)$ is given by

$$\{(m - n + k(m + 2))\Lambda_0 + (n - k(m + 2))\Lambda_1$$

$$| \; 0 \leq n \leq t + 2u - 2, \; 0 \leq k \leq u - 1 \; (n, k \in \mathbf{Z}) \,\}.$$

In this paper, we concentrate on the level $-1/2$ case. The level $-1/2$ admissible weights are as follows.

$$\lambda_1 = -\frac{1}{2}\Lambda_0, \; \lambda_2 = -\frac{3}{2}\Lambda_0 + \Lambda_1, \; \lambda_3 = -\frac{1}{2}\Lambda_1, \; \lambda_4 = \Lambda_0 - \frac{3}{2}\Lambda_1.$$

In the level $-1/2$ case, the characters of the irreducible highest weight $\widehat{sl}_2$-modules $L(\lambda)$ have the following formulas due to [14].

$$\text{ch}\, L(\lambda_1) + p^{\frac{1}{2}}\text{ch}\, L(\lambda_2) = \frac{e^{-\frac{1}{2}\Lambda_0}}{(p^{\frac{1}{2}}z^{\frac{1}{2}}; p)_\infty (p^{\frac{1}{2}}z^{-\frac{1}{2}}; p)_\infty} \; ,$$

$$\text{ch}\, L(\lambda_3) + \text{ch}\, L(\lambda_4) = \frac{e^{-\frac{1}{2}\Lambda_1}}{(pz^{-\frac{1}{2}}; p)_\infty (z^{\frac{1}{2}}; p)_\infty} \; ,$$

where $p = e^{-\delta}$, $z = e^{-\alpha_1}$ and $(a; p)_\infty = \prod_{n=0}^\infty (1 - ap^n)$.

We use the difference operator $_{q^{\frac{1}{2}}}\partial_z$ defined by

$$_{q^{\frac{1}{2}}}\partial_z f(z) := \frac{f(q^{\frac{1}{2}}z) - f(q^{-\frac{1}{2}}z)}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})z}.$$

3 Construction of representation

Let $\{a_n, b_n | n \in \mathbf{Z}\}$ be a set of operators satisfying the following commutation relations for $n \neq 0$.

$$[a_n, a_{-n}] = \frac{[2n][-\frac{1}{2}n]}{n} \quad , \quad [b_n, b_{-n}] = n.$$

The other commutation relations are zero. We define the Fock module $\mathcal{F}_{l_1, l_2}$ by the defining relations

$$a_n|l_1, l_2\rangle = 0 \quad (n > 0) , \quad b_n|l_1, l_2\rangle = 0 \quad (n > 0) ,$$

$$a_0|l_1, l_2\rangle = l_1|l_1, l_2\rangle , \quad b_0|l_1, l_2\rangle = l_2|l_1, l_2\rangle ,$$

where $|l_1, l_2\rangle$ is the vacuum vector. The grading operator $\overline{d}$ is defined by

$$\overline{d}.f|l_1, l_2\rangle = (-\sum_{l=1}^{k} m_l - \sum_{l=1}^{k'} n_l + \frac{l_1^2 - l_2^2 + l_2}{2})f|l_1, l_2\rangle$$

for $f = a_{-m_1} \cdots a_{-m_k} b_{-n_1} \cdots b_{-n_{k'}}$. We set

$$\widetilde{\mathcal{F}} := \bigoplus_{l_1 \in \frac{1}{2}\mathbf{Z}, l_2 \in \mathbf{Z}} \mathcal{F}_{l_1, l_2}.$$

We define the operators e^{P_a} and e^{P_b} on $\widetilde{\mathcal{F}}$ as follows.

$$e^{P_a}|l_1, l_2\rangle = |l_1 + 1, l_2\rangle \quad , \quad e^{P_b}|l_1, l_2\rangle = |l_1, l_2 + 1\rangle.$$

Let $:\quad:$ be the normal ordering defined by

$$: a_m a_n := a_m a_n \ (m \leq n), \ a_n a_m \ (m > n),$$

$$: b_m b_n := b_m b_n \ (m \leq n), \ b_n b_m \ (m > n),$$

$$: e^{P_a} a_0 :=: a_0 e^{P_a} := e^{P_a} a_0 ,$$

$$: e^{P_b} b_0 :=: b_0 e^{P_b} := e^{P_b} b_0 .$$

Consider the following currents.

$$Y_a^{\pm}(z) = \exp(\pm \sum_{k=1}^{\infty} \frac{a_{-k}}{[-\frac{1}{2}k]} q^{\pm \frac{k}{4}} z^k) \exp(\mp \sum_{k=1}^{\infty} \frac{a_k}{[-\frac{1}{2}k]} q^{\pm \frac{k}{4}} z^{-k}) e^{\pm 2P_a} z^{\mp 2a_0},$$

$$Y_b^{\pm}(z) = \exp(\pm \sum_{k=1}^{\infty} \frac{b_{-k}}{k} z^k) \exp(\mp \sum_{k=1}^{\infty} \frac{b_k}{k} z^{-k}) e^{\pm P_b} z^{\pm b_0}.$$

Proposition 3.1. $\widetilde{\mathcal{F}}$ *is a U_q-module of level $-\frac{1}{2}$ under the action of U_q defined by*

$$\gamma \longmapsto q^{-\frac{1}{2}} \quad , \quad K \longmapsto q^{a_0} \quad ,$$

$$h_n \longmapsto a_n \quad , \quad q^d \longmapsto q^{\overline{d}} \quad ,$$

$$X^+(z) \longmapsto \left(\frac{1}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} : Y_b^+(z)_{q^{\frac{1}{2}}} \partial_z^2 Y_b^+(z) : \right.$$

$$\left. - :_{q^{\frac{1}{2}}} \partial_z Y_b^+(q^{\frac{1}{2}}z)_{q^{\frac{1}{2}}} \partial_z Y_b^+(q^{-\frac{1}{2}}z) : \right) Y_a^+(z) ,$$

$$X^-(z) \longmapsto \frac{1}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} : Y_b^-(q^{\frac{1}{2}}z) Y_b^-(q^{-\frac{1}{2}}z) : Y_a^-(z)$$

where

$$X^{\pm}(z) := \sum_{k \in Z} x_k^{\pm} z^{-k-1}.$$

The proof will be given in Section 5.1.

The weight of $|l_1, l_2\rangle$ is $\frac{1}{2} l_1 \alpha_1 - \frac{1}{2}(l_1^2 - l_2^2 + l_2)\delta$. The space $\widetilde{\mathcal{F}}$ includes four irreducible highest weight submodules as we will see below. Consider the operator defined by

$$Q^- := \oint Y_b^-(z) \frac{dz}{2\pi\sqrt{-1}} \quad : \quad \mathcal{F}_{l_1, l_2} \longrightarrow \mathcal{F}_{l_1, l_2 - 1}.$$

We define $\mathcal{F}_i$ $(i = 1, 2, 3, 4)$ by

$$\mathcal{F}_i := \bigoplus_{l \in 2\mathbf{Z}} \mathrm{Ker}\,(Q^- : \mathcal{F}_{l+r_i, l+t_i} \to \mathcal{F}_{l+r_i, l+t_i-1}).$$

where $(r_i, t_i) = (0, 0), (1, 1), (-1/2, 0), (1/2, 1)$ for $i = 1, 2, 3, 4$, respectively. Then we have the following theorem.

Theorem 3.2. *Each $\mathcal{F}_i$ $(i = 1, 2, 3, 4)$ is an irreducible highest weight U_q-module isomorphic to $V(\lambda_i')$, where $\lambda_1' = -\frac{1}{2}\Lambda_0$, $\lambda_2' = -\frac{3}{2}\Lambda_0 + \Lambda_1 - \frac{1}{2}\delta$,*

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$\lambda'_3 = -\frac{1}{2}\Lambda_1 + \frac{1}{8}\delta$, $\lambda'_4 = \Lambda_0 - \frac{3}{2}\Lambda_1 + \frac{1}{8}\delta$. *The highest weight vectors are given by* $|\lambda'_1\rangle = |0, 0\rangle$, $|\lambda'_2\rangle = b_{-1}|1, 1\rangle$, $|\lambda'_3\rangle = |-1/2, 0\rangle$, $|\lambda'_4\rangle = |-3/2, -1\rangle$.

Proof. It can be checked immediately that each $|\lambda'_i\rangle$ is a weight vector of weight λ'_i, satisfies the highest weight condition and belongs to $\mathcal{F}_i$. Since $[U_q, Q^-] = 0$ (which will be proved in Section 5.2), $\mathcal{F}_i$ is a U_q-module. Next, we calculate the character of $\mathcal{F}_i$. We may understand Q^- as the zero mode η_0 of the fermionic ghost system (ξ, η) of dimension $(0, 1)$.

$$\xi(z) = Y_b^+(z) = \sum_{n \in \mathbf{Z}} \xi_n z^{-n}, \quad \eta(z) = Y_b^-(z) = \sum_{n \in \mathbf{Z}} \eta_n z^{-n-1}.$$

Since we have $\eta_0^2 = 0$ and $\xi_0\eta_0 + \eta_0\xi_0 = 1$, we obtain the following exact sequence for any l_1, l_2

$$0 \longrightarrow \mathrm{Ker}\,_{\mathcal{F}_{l_1,l_2}} Q^- \longrightarrow \mathcal{F}_{l_1,l_2} \xrightarrow{Q^-} \mathcal{F}_{l_1,l_2-1} \xrightarrow{Q^-} \mathcal{F}_{l_1,l_2-2} \xrightarrow{Q^-} \cdots.$$

Using this exact sequence, we can compute the character of $\mathcal{F}_i$. Since a_n and b_n have weight $n\delta$, we can easily see

$$\mathrm{ch}\,\mathcal{F}_{l_1,l_2} = \frac{z^{-\frac{1}{2}l_1}p^{-\frac{1}{2}(l_1^2 - l_2^2 + l_2)}}{(p;p)_\infty^2},$$

and hence

$$\mathrm{ch}\,(\mathcal{F}_1 \oplus \mathcal{F}_2) = \sum_{m \in \mathbf{Z}} \mathrm{ch}\,(\mathrm{Ker}\,_{\mathcal{F}_{m,m}} Q^-)$$

$$= (p;p)_\infty^{-2} \sum_{m \in \mathbf{Z}} z^{-\frac{1}{2}m} \sum_{k \leq m} (-1)^{m-k} p^{-\frac{1}{2}(m^2 - k^2 + k)}$$

$$\overset{(*)}{=} \frac{1}{(p^{\frac{1}{2}}z^{\frac{1}{2}};p)_\infty (p^{\frac{1}{2}}z^{-\frac{1}{2}};p)_\infty}.$$

The last equality $(*)$ will be proved in Section 5.3. Similarly, we have

$$\mathrm{ch}\,(\mathcal{F}_3 \oplus \mathcal{F}_4) = \frac{p^{-\frac{1}{8}}z^{\frac{1}{4}}}{(pz^{\frac{1}{2}};p)_\infty (z^{-\frac{1}{2}};p)_\infty}.$$

Hence, by the character formulas in Section 2,

$$\mathrm{ch}\,\mathcal{F}_i = \mathrm{ch}\,L(\lambda'_i).$$

As in Lusztig([15]), $\mathcal{F}_i$ becomes a certain $\widehat{sl}_2$-module in the classical limit $q \to 1$. Since the dimension of each weight space is invariant in the limit, the $\widehat{sl}_2$-module in the classical limit is irreducible. Therefore $\mathcal{F}_i$ is irreducible for a generic q. $\qquad\square$

We obtain $V(\lambda_i)$ from $V(\lambda_i')$ by shifting the grading operator d to d_i, where $d_i = d$, $d + \frac{1}{2}$, $d - \frac{1}{8}$, $d - \frac{1}{8}$, for $i = 1, 2, 3, 4$, respectively. This representation becomes the representation constructed by Feingold and Frenkel in [5] in the classical limit $q \to 1$. The relation between a_n, b_n and $\phi_1(z)$, $\phi_2(z)$ in the introduction is as follows.

$$ Y_a^{\pm}(z) \longrightarrow : e^{\pm 2\phi_1(z)} : \ , \quad Y_b^{\pm}(z) \longrightarrow : e^{\pm \phi_2(z)} : \ . $$

4   The q-vertex operators

4.1   Definition of the q-vertex operators

We recall the definition and some properties of the q-vertex operators in our case([7],[1]). We consider the 2-dimensional U_q' module $V = \mathbf{K}v_+ \oplus \mathbf{K}v_-$. The U_q'-module structure on V is given by

$$ e_1.v_+ = f_0.v_+ = 0, \ e_1.v_- = f_0.v_- = v_+, $$

$$ e_0.v_- = f_1.v_- = 0, \ e_0.v_+ = f_1.v_+ = v_-, $$

$$ t_0.v_{\pm} = q^{\mp 1}v_{\pm}, \ t_1.v_{\pm} = q^{\pm 1}v_{\pm}. $$

The affinization of V is the following U_q-module V_z.

$$ V_z := V \otimes \mathbf{K}[z, z^{-1}]. $$

We define the U_q-module structure on V_z as follows.

$$ e_i.(v \otimes z^m) = e_i.v \otimes z^{m+\delta_{i0}} \quad , \quad f_i.(v \otimes z^m) = f_i.v \otimes z^{m-\delta_{i0}} $$

$$ t_i.(v \otimes z^m) = t_i.v \otimes z^m \quad , \quad q^d.(v \otimes z^m) = q^m v \otimes z^m. $$

Definition 4.1.([7]) *The q-vertex operator is a U_q-homomorphism of one of the following types.*

Type I :

$$\tilde{\Phi}_\lambda^{\mu V}(z) : V(\lambda) \longrightarrow V(\mu)\widehat{\otimes}V_z$$

Type II :

$$^V\tilde{\Phi}_\lambda^{\mu}(z) : V(\lambda) \longrightarrow V_z\widehat{\otimes}V(\mu)$$

□

The existence conditions of the q-vertex operators are known in [1]. In our case, the q-vertex operators exist only for $(\lambda, \mu) = (\lambda_1, \lambda_2), (\lambda_2, \lambda_1), (\lambda_3, \lambda_4), (\lambda_4, \lambda_3)$. Furthermore, each of them is unique up to a scalar. Here, we take the following normalization.

$$\tilde{\Phi}_\lambda^{\mu V}(z)|\lambda\rangle = |\mu\rangle \otimes v_+ + (\text{ the terms of positive powers in } z)$$

$$\text{for } (\lambda, \mu) = (\lambda_2, \lambda_1), (\lambda_3, \lambda_4),$$

$$\tilde{\Phi}_\lambda^{\mu V}(z)|\lambda\rangle = |\mu\rangle \otimes v_- + (\text{ the terms of positive powers in } z)$$

$$\text{for } (\lambda, \mu) = (\lambda_1, \lambda_2), (\lambda_4, \lambda_3).$$

For the type II, we take a similar normalization.

4.2   Bosonizations

In this subsection, we construct an explicit form of the q-vertex operators on $\mathcal{F}_i$'s. We define the components $\tilde{\Phi}_\lambda^{\mu V}{}_\pm(z)$ of the q-vertex operators as follows.

$$\tilde{\Phi}_\lambda^{\mu V}(z)|u\rangle = \tilde{\Phi}_\lambda^{\mu V}{}_+(z)|u\rangle \otimes v_+ + \tilde{\Phi}_\lambda^{\mu V}{}_-(z)|u\rangle \otimes v_- \qquad \text{for} \quad |u\rangle \in V(\lambda),$$

For the type II, the components are defined similarly.

Using the same technique as in [9] (c.f. [11],[10]), the components of the q-vertex operators are given by the following theorem.

Theorem 4.2.

1) $\tilde{\Phi}_\lambda^{\mu V}{}_-(z) = \mathcal{J}^+(q^{\frac{3}{2}}z)_{q^{\frac{1}{2}}}\partial_z Y_b^+(q^{\frac{3}{2}}z)\,r,$

$\tilde{\Phi}_\lambda^{\mu V}{}_+(z) = [\ \tilde{\Phi}_\lambda^{\mu V}{}_-(z)\ ,\ f_1\ ]_q\ ,$

$r = 1, -q^{\frac{3}{2}}z, -q^{\frac{3}{4}}z^{\frac{1}{2}}, -q^{\frac{3}{4}}z^{\frac{1}{2}}$

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$$for \ (\lambda, \mu) = (\lambda_1, \lambda_2), (\lambda_2, \lambda_1), (\lambda_3, \lambda_4), (\lambda_4, \lambda_3),$$

2) $\ {}^V \widetilde{\Phi}^{\mu}_{\lambda+}(z) = \mathcal{J}^-(q^{-\frac{1}{2}}z)Y_b^-(q^{-\frac{1}{2}}z) \, r,$

$${}^V \widetilde{\Phi}^{\mu}_{\lambda-}(z) = [\ {}^V \widetilde{\Phi}^{\mu}_{\lambda+}(z) \ , e_1 \ ]_q \ ,$$

$$r = -q^{-1}, q^{-\frac{1}{2}}z, q^{-\frac{1}{4}}z^{\frac{1}{2}}, q^{-\frac{5}{4}}z^{\frac{1}{2}}$$

$$for \ (\lambda, \mu) = (\lambda_1, \lambda_2), (\lambda_2, \lambda_1), (\lambda_3, \lambda_4), (\lambda_4, \lambda_3),$$

where

$$[X, Y]_q = XY - qYX,$$

$$\mathcal{J}^{\pm}(z) = \exp(\pm \sum_{k=1}^{\infty} \widetilde{a}_{-k} q^{\mp \frac{1}{4}k} z^k) \exp(\pm \sum_{k=1}^{\infty} \widetilde{a}_k q^{\mp \frac{1}{4}k} z^{-k}) e^{\pm P_a} z^{\mp a_0},$$

$$\widetilde{a}_k = \frac{q^{\frac{1}{2}k} + q^{-\frac{1}{2}k}}{[2k]} a_k.$$

Proof. The proof of the intertwining relations will be given in Section 5.4. It can be checked directly that the operators defined in 1) and 2) anticommute with Q^-. Therefore, they give operators on the $\mathcal{F}_i$'s. □

We give an example of the two-point functions of the q-vertex operators by using the above bosonization formulas.

$$\langle \lambda_1 | \widetilde{\Phi}^V(z_2) \widetilde{\Phi}^V(z_1) | \lambda_1 \rangle = \frac{(q^3 z; q^4)_{\infty}(q^6 z; q^4)_{\infty}}{(qz; q^4)_{\infty}(q^4 z; q^4)_{\infty}}(v_+ \otimes v_- - qv_- \otimes v_+),$$

where $z = z_1/z_2$.

5 Proofs

In this section, we give the proofs postponed in the previous sections. We list the following formulas on $Y_a^{\pm}(z)$, $Y_b^{\pm}(z)$ and $\mathcal{J}^{\pm}(z)$ defined in the previous sections.

(1) $\ Y_a^{\pm}(z)Y_a^{\pm}(w) = \dfrac{: Y_a^{\pm}(z)Y_a^{\pm}(w) :}{(z - q^{\pm 2}w)(z - qw)(z - w)(z - q^{-1}w)},$

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(2) $\quad Y_a^{\pm}(z)Y_a^{\mp}(w)$

$$=: Y_a^{\pm}(z)Y_a^{\mp}(w): (z - q^{\frac{3}{2}}w)(z - q^{\frac{1}{2}}w)(z - q^{-\frac{1}{2}}w)(z - q^{-\frac{3}{2}}w),$$

(3) $\quad Y_b^{\pm}(z)Y_b^{\pm}(w) =: Y_b^{\pm}(z)Y_b^{\pm}(w): (z - w),$

(4) $\quad Y_b^{\pm}(z)Y_b^{\mp}(w) = \dfrac{: Y_b^{\pm}(z)Y_b^{\mp}(w) :}{z - w},$

(5) $\quad \mathcal{J}^{\pm}(z)Y_a^{\pm}(w) =: \mathcal{J}^{\pm}(z)Y_a^{\pm}(w): \dfrac{1}{(z - q^{\frac{1}{2}}w)(z - q^{-\frac{1}{2}}w)},$

(6) $\quad Y_a^{\pm}(w)\mathcal{J}^{\pm}(z) =: \mathcal{J}^{\pm}(z)Y_a^{\pm}(w): \dfrac{1}{(w - q^{\frac{1}{2}}z)(w - q^{-\frac{1}{2}}z)},$

(7) $\quad \mathcal{J}^{\pm}(z)Y_a^{\mp}(w) =: \mathcal{J}^{\pm}(z)Y_a^{\mp}(w): (z - w)(z - q^{\mp 1}w),$

(8) $\quad Y_a^{\mp}(w)\mathcal{J}^{\pm}(z) =: \mathcal{J}^{\pm}(z)Y_a^{\mp}(w): (w - z)(w - q^{\mp 1}z),$

where $\frac{1}{z-w}$ means the formal power series $\frac{1}{z}\sum_{k\geq 0}(\frac{w}{z})^k$. The above equations should be regarded as formal power series in z and w. In the proofs, we will use the formal delta function $\delta(z) = \sum_{k \in \mathbf{Z}} z^k$.

5.1 Proof of Proposition 3.1.

Before starting the proof, we recall the defining relations of U_q in terms of the generating functions $X^{\pm}(z) = \sum_{k \in \mathbf{Z}} x_k^{\pm} z^{-k-1}$ in the level $-1/2$ case.

R1) $\quad KK^{-1} = K^{-1}K = 1, \quad q^d q^{-d} = q^{-d}q^d = 1, \quad Kq^d = q^d K,$

R2) $\quad q^d h_k q^{-d} = q^k h_k, \quad q^d X^{\pm}(z)q^{-d} = q^{-1}X^{\pm}(q^{-1}z),$

R3) $\quad [h_k, K] = 0, \quad [h_k, h_l] = \delta_{k+l,0}\dfrac{[2k][-\frac{1}{2}k]}{k},$

R4) $\quad KX^{\pm}(z)K^{-1} = q^{\pm 2}X^{\pm}(z), \quad [a_k, X^{\pm}(z)] = \pm\dfrac{[2k]}{k}q^{\pm\frac{|k|}{4}}X^{\pm}(z),$

R5) $\quad (z - q^{-2}w)X^{-}(z)X^{-}(w) + (w - q^{-2}z)X^{-}(w)X^{-}(z) = 0 ,$

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R6) $(z - q^2 w)X^+(z)X^+(w) + (w - q^2 z)X^+(w)X^+(z) = 0$,

R7) $[X^+(z), X^-(w)]$

$$= \frac{1}{q - q^{-1}} \frac{1}{zw} \left(\delta(\frac{zq^{\frac{1}{2}}}{w})\psi(zq^{\frac{1}{4}}) - \delta(\frac{z}{wq^{\frac{1}{2}}})\varphi(w^{-1}q^{-\frac{1}{4}}) \right)$$

where

$$\psi(z) = K \exp\left((q - q^{-1}) \sum_{k=1}^{\infty} h_k z^{-k} \right) ,$$
$$\varphi(z) = K^{-1} \exp\left(-(q - q^{-1}) \sum_{k=1}^{\infty} h_{-k} z^{-k} \right) .$$

The relations $R1)$, $R2)$ and $R3)$ are trivial. The relation $R4)$ follows from the following formulas.

$$q^{a_0} Y_a^{\pm}(z) q^{-a_0} = q^{\pm 2} Y_a^{\pm}(z),$$

$$[a_k, Y_a^{\pm}(z)] = \pm \frac{[2k]}{k} q^{\pm \frac{|k|}{4}} z^k Y_a^{\pm}(z) \qquad \text{for } k \neq 0.$$

For the proof of relations $R5)$, $R6)$ and $R7)$, we introduce $M_1^+(z)$, $M_2^+(z)$, $M_3^+(z)$ and $M^{\pm}(z)$ as follows.

$$M_1^+(z) =: Y_a^+(z)Y_b^+(qz)Y_b^+(z) :,$$

$$M_2^+(z) =: Y_a^+(z)Y_b^+(z)Y_b^+(q^{-1}z) :,$$

$$M_3^+(z) =: Y_a^+(z)Y_b^+(qz)Y_b^+(q^{-1}z) :,$$

$$M^+(z) = q^{\frac{1}{2}} M_1^+(z) + q^{-\frac{1}{2}} M_2^+(z) - (q^{\frac{1}{2}} + q^{-\frac{1}{2}})M_3^+(z),$$

$$M^-(z) =: Y_a^+(z)Y_b^-(q^{\frac{1}{2}}z)Y_b^-(q^{-\frac{1}{2}}z) : .$$

The action of $X^{\pm}(z)$ is expressed by these currents as follows.

$$X^+(z) \longmapsto -\frac{M^+(z)}{(q - q^{-1})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})z^2} , \qquad X^-(z) \longmapsto \frac{M^-(z)}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}.$$

Using $(1) - (4)$, we see

$$(z - q^{-2}w)M^-(z)M^-(w) =: M^-(z)M^-(w) : (z - w).$$

The relation $R5$) follows from this.

Next, we check the relation $R6$).

$$(z - q^2 w)M^+(z)M^+(w)$$

$$=: M_1^+(z)M_1^+(w) : q^3(z - w)$$

$$+ : M_1^+(z)M_2^+(w) : q^2 \frac{(z - q^{-1}w)(z - q^{-2}w)}{z - qw}$$

$$+ : M_1^+(z)M_3^+(w) : q^{\frac{5}{2}}(q^{\frac{1}{2}} + q^{-\frac{1}{2}})(z - q^{-2}w)$$

$$+ : M_2^+(z)M_1^+(w) : q^{-2}\frac{(z - qw)(z - q^2 w)}{z - q^{-1}w}$$

$$+ : M_2^+(z)M_2^+(w) : q^{-3}(z - w)$$

$$+ : M_2^+(z)M_3^+(w) : q^{-\frac{5}{2}}(q^{\frac{1}{2}} + q^{-\frac{1}{2}})(z - q^2 w)$$

$$+ : M_3^+(z)M_1^+(w) : q^{\frac{1}{2}}(q^{\frac{1}{2}} + q^{-\frac{1}{2}})(z - q^2 w)$$

$$+ : M_3^+(z)M_2^+(w) : q^{-\frac{1}{2}}(q^{\frac{1}{2}} + q^{-\frac{1}{2}})(z - q^{-2}w)$$

$$+ : M_3^+(z)M_3^+(w) : (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^2 \frac{(z - w)(z - q^2 w)(z - q^{-2}w)}{(z - qw)(z - q^{-1}w)}.$$

Symmetrizing it with respect to z and w, we have

$$(z - q^2 w)M^+(z)M^+(w) + (w - q^2 z)M^+(w)M^+(z)$$

$$=: M_1^+(z)M_2^+(w) : q(z - q^{-1}w)(z - q^{-2}w)\frac{1}{w}\delta(\frac{qw}{z})$$

$$+ : M_2^+(z)M_1^+(w) : q^{-1}(z - qw)(z - q^2 w)\frac{1}{w}\delta(\frac{w}{qz})$$

$$+ : M_3^+(z)M_3^+(w) :$$

$$\times \frac{(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^2(z - w)(z - q^2 w)(z - q^{-2}w)\left(\delta(\frac{qw}{z}) - \delta(\frac{qz}{w})\right)}{zw(q - q^{-1})}$$

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$$= \left(: M_1^+(qw)M_2^+(w) : - : M_3^+(qw)M_3^+(w) : \right)$$

$$\times q(q^{-1}-q)(q-q^{-2})w\delta(\frac{qw}{z})$$

$$+ \left(: M_2^+(q^{-1}w)M_1^+(w) : - : M_3^+(q^{-1}w)M_3^+(w) : \right)$$

$$\times q^{-1}(q^{-1}-q)(q^{-1}-q^2)w\delta(\frac{w}{qz})$$

$$= 0.$$

The relation $R6)$ follows from this.

Finally, we check the relation $R7)$.

$$[M_1^+(z), M^-(w)]$$

$$=: M_1^+(z)M^-(w) : q^{-2}\frac{z-q^{\frac{3}{2}}w}{z-q^{-\frac{1}{2}}w} - : M_1^+(z)M^-(w) : \frac{w-q^{-\frac{3}{2}}z}{w-q^{\frac{1}{2}}z}$$

$$=: M_1^+(z)M^-(w) : (z-q^{-\frac{3}{2}}w)\delta(\frac{zq^{\frac{1}{2}}}{w})$$

$$= (q^{-2}-1)q^{a_0}\exp\left((q-q^{-1})\sum_{n=1}^{\infty}a_n(q^{\frac{1}{4}}z)^{-n}\right)\delta(\frac{zq^{\frac{1}{2}}}{w}).$$

Similarly,

$$[M_2^+(z), M^-(w)]$$

$$= (q^2-1)q^{-a_0}\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}a_{-n}(q^{-\frac{1}{4}}w^{-1})^{-n}\right)\delta(\frac{z}{wq^{\frac{1}{2}}}),$$

$$[M_3^+(z), M^-(w)] = 0.$$

The relations $R7)$ follows from the above formulas. The proof is completed.

5.2 Proof of $[U_q, Q^-] = 0$

It is trivial that Q^- commutes with $\gamma^{\frac{1}{2}}$, K, h_k and x_k^-. We will show $[X^+(z), Q^-] = 0$. We need to calculate the commutator $[M_i^+(z), Y_b^-(w)]$.

$$[M_1^+(z), Y_b^-(w)] =: M_1^+(z)Y_b^-(w) : \frac{1}{zw(q-1)}\left(\delta(\frac{w}{z}) - \delta(\frac{qz}{w})\right).$$

$$[M_2^+(z), Y_b^-(w)] =: M_2^+(z)Y_b^-(w) : \frac{1}{zw(q^{-1}-1)}\left(\delta(\frac{w}{z}) - \delta(\frac{z}{qw})\right).$$

$$[M_3^+(z), Y_b^-(w)] =: M_3^+(z)Y_b^-(w) : \frac{1}{zw(q-q^{-1})}\left(\delta(\frac{qw}{z}) - \delta(\frac{qz}{w})\right).$$

Using these formulas, we have

$$[M^+(z), Y_b^-(w)] = Y_a^+(z)\left(Y_b^+(qz)\left(\delta(\frac{w}{z}) - \delta(\frac{qw}{z})\right)\right.$$

$$\left. +Y_b^+(z)\left(\delta(\frac{qw}{z}) - \delta(\frac{w}{qz})\right) + Y_b^+(q^{-1}z)\left(\delta(\frac{w}{qz}) - \delta(\frac{w}{z})\right)\right)$$

Picking up the coefficient of w^{-1}, we have

$$[X^+(z), Q^-] = 0.$$

5.3 Proof of the identity in the proof of theorem 3.2.

In this subsection, we prove the following identity $(*)$ used in the calculation of the characters.

$$(*) \quad (p;p)_\infty^{-2} \sum_{m \in \mathbf{Z}} z^{-\frac{1}{2}m} \sum_{k \geq -m} (-1)^{k+m} p^{\frac{1}{2}(k^2-m^2+k)} = \frac{1}{(p^{\frac{1}{2}}z^{\frac{1}{2}};p)_\infty(p^{\frac{1}{2}}z^{-\frac{1}{2}};p)_\infty}.$$

In order to prove the above formula, it is sufficient to show

$$\sum_{n \in \mathbf{Z}}(-1)^n p^{\frac{1}{2}n^2}z^{\frac{1}{2}n}\sum_{m \in \mathbf{Z}}z^{-\frac{1}{2}m}\sum_{k \geq -m}(-1)^{k+m}p^{\frac{1}{2}(k^2+k-m^2)} = (p;p)_\infty^3.$$

The left hand side is rewritten as

$$\sum_{l \in \mathbf{Z}}(-1)^l p^{\frac{1}{2}l^2}z^{\frac{1}{2}l}\sum_{m \in \mathbf{Z}}p^{ml}\sum_{k \geq -m}(-1)^k p^{\frac{1}{2}(k^2+k)}.$$

Now, we show $S_l = \sum_{m \in \mathbf{Z}} p^{ml} \sum_{k \geq -m} (-1)^k p^{\frac{1}{2}(k^2+k)} = (p;p)_\infty^3 \delta_{l,0}$.

$$(1 - p^l)S_l = S_l - \sum_{m \in \mathbf{Z}} p^{(m+1)l} \sum_{k \geq -m} (-1)^k p^{\frac{1}{2}(k^2+k)}$$

$$= S_l - \sum_{m \in \mathbf{Z}} p^{ml} \sum_{k \geq -m+1} (-1)^k p^{\frac{1}{2}(k^2+k)}$$

$$= \sum_{m \in \mathbf{Z}} p^{ml} (-1)^m p^{\frac{1}{2}(m^2-m)}$$

$$= (p;p)_\infty (p^l;p)_\infty (p^{1-l};p)_\infty$$

$$= 0.$$

Hence, $S_l = 0$ for $l \neq 0$.

$$S_0 = \sum_{m \in \mathbf{Z}} \sum_{k \geq -m} (-1)^k p^{\frac{1}{2}(k^2+k)} = \sum_{m \in \mathbf{Z}} \sum_{k \geq |m|} (-1)^k p^{\frac{1}{2}(k^2+k)}$$

$$= \sum_{n \geq 0} (-1)^n (2n+1) p^{\frac{1}{2}(n^2+n)} = (p;p)_\infty^3.$$

The identity was proved.

5.4   Proof of Theorem 4.2.

We give the proof for type I. The intertwining conditions with the Chevalley generators are as follows.

$V1)$ $\quad t_1 \widetilde{\Phi}_\pm^V(z) t_1^{-1} = q^{\mp 1} \widetilde{\Phi}_\pm^V(z)$

$V2)$ $\quad t_0 \widetilde{\Phi}_\pm^V(z) t_0^{-1} = q^{\pm 1} \widetilde{\Phi}_\pm^V(z)$

$V3)$ $\quad [\widetilde{\Phi}_+^V(z), e_0] = 0$

$V4)$ $\quad [\widetilde{\Phi}_-^V(z), e_0] = z t_0 \widetilde{\Phi}_+^V(z)$

$V5)$ $\quad [\widetilde{\Phi}_+^V(z), e_1] = t_1 \widetilde{\Phi}_-^V(z)$

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$V6)$ $\quad [\widetilde{\Phi}^V_-(z), e_1] = 0$

$V7)$ $\quad [\widetilde{\Phi}^V_+(z), f_0]_q = z^{-1}\widetilde{\Phi}^V_-(z)$

$V8)$ $\quad [\widetilde{\Phi}^V_-(z), f_0]_{q^{-1}} = 0$

$V9)$ $\quad [\widetilde{\Phi}^V_-(z), f_1]_q = \widetilde{\Phi}^V_+(z)$

$V10)$ $\quad [\widetilde{\Phi}^V_+(z), f_1]_{q^{-1}} = 0$

where $[X, Y]_{q^{\pm 1}} = XY - q^{\pm 1}YX$.

These conditions are not independent. Actually, as we will see below, some of them follow from the others. We recall our construction of $\widetilde{\Phi}^V_\pm(z)$.

$$\widetilde{\Phi}^V_-(z) = \mathcal{J}^+(q^{\frac{3}{2}}z)_{q^{\frac{1}{2}}}\partial_z Y^+_b(q^{\frac{3}{2}}z),$$

and $\widetilde{\Phi}^V_+(z)$ is given by $V9)$. From the construction, $V1)$, $V2)$ and $V9)$ are trivial. First, we shall reduce the conditions to $V3)$ and the following two equations.

(A) $\quad [\widetilde{\Phi}^V_-(z), X^+(w)] = 0.$

(B) $\quad [\widetilde{\Phi}^V_-(z), X^-(w)]_q = [\widetilde{\Phi}^V_-(z), X^-(w)]_{q^{-1}}\left(\dfrac{w}{q^{\frac{1}{2}}z}\right).$

It can be checked immediately that $V6)$ and $V8)$ follow from (A) and that $V4)$ follows from (B).

$\quad V5)$ follows from $V1)$, $V6)$ and $V9)$.

$$[\widetilde{\Phi}^V_+(z), e_1]$$

$$= [[\widetilde{\Phi}^V_-(z), f_1]_q, e_1]$$

$$= [[\widetilde{\Phi}^V_-(z), e_1], f_1]_q - [\widetilde{\Phi}^V_-(z), [e_1, f_1]]_q$$

$$= -[\widetilde{\Phi}^V_-(z), [e_1, f_1]]_q$$

$$= -[\widetilde{\Phi}^V_-(z), \dfrac{t_1 - t_1^{-1}}{q - q^{-1}}]_q$$

$$= t_1 \widetilde{\Phi}^V_-(z).$$

$V7$) follows from $V2$), $V4$) and $V8$).

$$[\widetilde{\Phi}^V_+(z), f_0]_q$$

$$= [z^{-1} t_0^{-1} [\widetilde{\Phi}^V_-(z), e_0], f_0]_q$$

$$= z^{-1} t_0^{-1} [[\widetilde{\Phi}^V_-(z), e_0], f_0]_{q^{-1}}$$

$$= z^{-1} t_0^{-1} ([[\widetilde{\Phi}^V_-(z), f_0]_{q^{-1}}, e_0]_q + [\widetilde{\Phi}^V_-(z), [e_0, f_0]]_{q^{-1}})$$

$$= z^{-1} t_0^{-1} [\widetilde{\Phi}^V_-(z), [e_0, f_0]]_{q^{-1}}$$

$$= z^{-1} t_0^{-1} [\widetilde{\Phi}^V_-(z), \frac{t_0 - t_0^{-1}}{q - q^{-1}}]_{q^{-1}}$$

$$= z^{-1} \widetilde{\Phi}^V_-(z).$$

$V10$) follows from $V4$), $V9$) and $V3$).

$$[\widetilde{\Phi}^V_+(z), f_1]_{q^{-1}}$$

$$= [z^{-1} t_0^{-1} [\widetilde{\Phi}^V_-(z), e_0], f_1]_{q^{-1}}$$

$$= z^{-1} t_0^{-1} [[\widetilde{\Phi}^V_-(z), e_0], f_1]_q$$

$$= z^{-1} t_0^{-1} [[\widetilde{\Phi}^V_-(z), f_1]_q, e_0]$$

$$= z^{-1} t_0^{-1} [\widetilde{\Phi}^V_+(z), e_0]$$

$$= 0.$$

Therefore, it remains to prove $V3$), (A) and (B).

Frist, we prove (B). Using $(4) - (8)$, we have

$$[\widetilde{\Phi}^V_-(z), X^-(w)]_q \times q^{\frac{3}{2}} z = - : Y^-_b(q^2 z) \mathcal{J}^+(q^{\frac{3}{2}} z) M^-(w) : \frac{w}{(w - q^{\frac{5}{2}} z)}$$

$$- : Y_b^-(qz)\mathcal{J}^+(q^{\frac{3}{2}}z)M^-(w) : \frac{q^{-\frac{1}{2}}w}{(z - q^{-\frac{1}{2}}w)},$$

$$[\widetilde{\Phi}_-^V(z), X^-(w)]_{q^{-1}} \times q^{\frac{3}{2}}z = - : Y_b^-(q^2z)\mathcal{J}^+(q^{\frac{3}{2}}z)M^-(w) : \frac{q^{\frac{1}{2}}z}{(w - q^{\frac{5}{2}}z)}$$

$$- : Y_b^-(qz)\mathcal{J}^+(q^{\frac{3}{2}}z)X^-(w) : \frac{z}{(z - q^{-\frac{1}{2}}w)}.$$

(B) follows from these two formulas.

Next, we prove (A).

$$[\widetilde{\Phi}_-^V(z), M_1^+(w)]$$

$$= - : \mathcal{J}^+(q^{\frac{3}{2}}z)Y_b^+(qz)M_1^+(w) : \frac{q(z - w)}{(q^2 - q)zw}\delta(\frac{q^2z}{w})$$

$$=: \mathcal{J}^+(q^{\frac{3}{2}}z)Y_b^+(qz)M_1^+(q^2z) : (q + 1)w^{-1}\delta(\frac{q^2z}{w}).$$

Similarly,

$$[\widetilde{\Phi}_-^V(z), M_2^+(w)]$$

$$=: \mathcal{J}^+(q^{\frac{3}{2}}z)Y_b^+(q^2z)M_2^+(qz) : (q + 1)w^{-1}\delta(\frac{qz}{w}),$$

$$[\widetilde{\Phi}_-^V(z), M_3^+(w)]$$

$$=: \mathcal{J}^+(q^{\frac{3}{2}}z)Y_b^+(q^2z)M_3^+(q^2z) : qw^{-1}\delta(\frac{q^2z}{w})$$

$$+ : \mathcal{J}^+(q^{\frac{3}{2}}z)Y_b^+(qz)M_3^+(qz) : w^{-1}\delta(\frac{qz}{w}).$$

From the above three formulas, we have

$$[\widetilde{\Phi}_-^V(z), M^+(w)] = 0.$$

We proved (A).

Finally, we prove $V3)$.

$$M^-(w_1)M^-(w_2)\widetilde{\Phi}_-^V(z)$$

$$-(q+q^{-1})M^-(w_1)\widetilde{\Phi}_-^V(z)M^-(w_2) + \widetilde{\Phi}_-^V(z)M^-(w_1)M^-(w_2)$$

$$=: \mathcal{J}^+(q^{\frac{3}{2}}z)Y_b^-(q^2z)M^-(w_1)M^-(w_2): \frac{q^{-4}(q^{\frac{1}{2}}+q^{-\frac{1}{2}})(w_1-w_2)}{(w_1-q^{\frac{5}{2}}z)(w_2-q^{\frac{5}{2}}z)}$$

$$-: \mathcal{J}^+(q^{\frac{3}{2}}z)Y_b^-(q^2z)M^-(w_1)M^-(w_2): \frac{q^{-1}(q^{\frac{1}{2}}+q^{-\frac{1}{2}})(w_1-w_2)}{(z-q^{-\frac{1}{2}}w_1)(z-q^{-\frac{1}{2}}w_2)}.$$

Since $: M^-(w_1)M^-(w_2) :=: M^-(w_2)M^-(w_1) :$, the above equation is anti-symmetric with respect to w_1 and w_2. Hence, the coefficient of $w_1^{-1}w_2^{-1}$ is zero. We proved $V3)$.

The proof for the type I q-vertex operators is completed. For type II, the proof is similar.

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